

# THE REPRESENTATION OF BOL ALGEBRAS

NDOUNE AND THOMAS B. BOUETO

**ABSTRACT.** We construct a representation of Bol algebras by using the notion of Bol modules.

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## 1. INTRODUCTION

It is well known that the algebraic systems which characterize locally a totally geodesic subspace is a Lie triple system [1], [2], [3], [4]. A Bol algebra is realized by equipping Lie triple System with an additional binary skew operation which satisfying a pseudo-differentiation property [5], [6]. More generally, the algebraic structures which characterize locally Bol loops are Bol algebras [7]. Till date, the representation of these algebras has not yet been given. And we know that, the representation of these algebras may help solve some problems in geometry and Physics. As far as geometry is concerned, this is used to construct the homology and the cohomology of Bol algebras and loops. As far as physics is concerned, it is used for the description of invariant properties of physical systems and the concomitant conservation laws as a result. J.Mostovoy and J.M.Prez-Izquierdo showed that, Malcev algebras and Lie triple systems are particular subclasses of Bol algebras [8]. The representation of Malcev algebras can be found in [9], and the Lie triple systems by T.L.Hodge and B.J.Parshall with a categorical approach [10], and the one done by W.Bertrand and M.Didry uses symmetric bundles of symmetric spaces[11]. A tangent algebra of a local analytic Moufang loop is a Malcev algebra. Now, there already exists representations of other classes of non-associative algebras; the case of alternative algebras was constructed by R.D.Schafer [14], and the one of Leibniz algebras by P. Kolesnikov [15]. The purpose of this paper is to present a construction of a representation of Bol algebras and some important properties. Our approach uses the notion of Bol modules defined in the paper, Ideals of Bol algebras introduced in [5] and the one of Lie triple systems [16]. This paper will be organized as

follow: first and fore, we will talk about Bol algebras, Next, a representation of these algebras will be presented. Dual representation and the duality principle will be introduced in the last section.

## 2. BOL ALGEBRAS

**Definition 2.0.1.** *A Bol algebra is a vector space  $\mathfrak{B}$  over a field  $K$  of which is closed with respect a trilinear operation  $(x; y, z)$  and with additional bilinear skew-symetric operation  $x \cdot y$  satisfying:*

- (i)  $(x; x, y) = 0$
- (ii)  $(x; y, z) + (z; x, y) + (y; z, x) = 0.$
- (iii)  $((x; y, z); \alpha, \beta) = ((x; \alpha, \beta); y, z) + (x; y; \alpha, \beta, z) + (x; y, (\alpha, \beta, z))$
- (iv)

$$(x \cdot y; \alpha, \beta) = (x; \alpha, \beta) \cdot y + x \cdot (y; \alpha, \beta) + (\alpha \cdot \beta; x, y) + (x \cdot y) \cdot (\alpha \cdot \beta)$$

for all  $x, y, z, \alpha$ , and  $\beta$  in  $\mathfrak{B}$ .

In other words, a Bol algebra is a lie triple system  $(\mathfrak{B}, (-; -, -))$  with an additional bilinear skew-symmetric operation  $x \cdot y$  such that, the derivation

$\mathfrak{D}_{\alpha, \beta} : x \longrightarrow (x; \alpha, \beta)$  on a ternary operation is a pseudo-differentiation with components  $\alpha, \beta$  on a binary operation, that is; for all  $x, y$  and  $z$  in  $\mathfrak{B}$ , we have

$$\mathfrak{D}_{\alpha, \beta}(x \cdot y) = (\mathfrak{D}_{\alpha, \beta}(x)) \cdot y + x \cdot (\mathfrak{D}_{\alpha, \beta}(y)) + (\alpha \cdot \beta; x, y) + (x \cdot y) \cdot (\alpha \cdot \beta).$$

$\mathfrak{D}_{\alpha, \beta}$  is differentiation on ternary operation  $(x; y, z)$  means that,

$$\mathfrak{D}_{\alpha, \beta}(x; y, w) = (\mathfrak{D}_{\alpha, \beta}(x); y, w) + (x; \mathfrak{D}_{\alpha, \beta}(y), w) + (x; y, \mathfrak{D}_{\alpha, \beta}(w)).$$

In fact, the Bol algebra defined above is called rigth Bol algebra [6].

Similarly, we define a left Bol algebra as a Lie triple system with an additional bilinear skew-symetric operation  $x \cdot y$  such that the derivation  $\Delta_{\alpha, \beta} : x \longrightarrow (\alpha; \beta, x)$  on a ternary operation is a pseudo-differentiation with components  $\alpha$  and  $\beta$ ; That is for all  $x, y, z$  in  $\mathfrak{B}$ ,

$$\Delta_{\alpha, \beta}(x \cdot y) = (\Delta_{\alpha, \beta}(x)) \cdot y + x \cdot (\Delta_{\alpha, \beta}(y)) + (\alpha \cdot \beta; x, y) + (\alpha \cdot \beta) \cdot (x \cdot y).$$

From a right Bol algebra  $\mathfrak{B}$  we can obtain a left Bol algebra  $\mathfrak{B}^{op}$  by considering  $\mathfrak{B}$  with the operations

$$[x; y, z] = -(z; x, y) \text{ and } [x, y] = -x \cdot y.$$

Therefore, in all what follows we will simply talk about Bol algebras.

We will note that Bol algebras can be realized as the tangent algebras to a Bol loops with the left Bol identity, and they allow embedding in Lie algebras [7], [12].

The subset  $S$  of a Bol algebra  $\mathfrak{B}$  is a sub-Bol algebra if it is closed under the ternary and the binary operations of  $\mathfrak{B}$ .

In all what follows, except otherwise stated, we will consider Bol algebras on a field  $K$  of characteristic zero.

**Definition 2.0.2.** A homomorphism  $\varphi : \mathfrak{B}_1 \rightarrow \mathfrak{B}_2$  between two Bol algebras is a linear map preserving the ternary and the binary operations.

Let us give some definitions and constructions on Bol algebras.

**Definition 2.0.3.** Given a Bol algebra  $(\mathfrak{B}, (-; -, -), \cdot)$  over a field  $K$  of characteristic zero, a pseudo-differentiation is a linear map  $D : \mathfrak{B} \rightarrow \mathfrak{B}$  for which, there exists  $z \in \mathfrak{B}$  (a companion of  $D$ ) such that  $D(x \cdot y) = D(x) \cdot y + x \cdot D(y) + (z; x, y) + (x \cdot y) \cdot z$ ; The companion is not necessarily unique.

The set of all companions of  $D$  is denoted  $Com(D)$ .

**Example 2.0.1.**  $\mathfrak{D}_{\alpha, \beta} : x \rightarrow (x; \alpha, \beta)$  is a pseudo-differentiation with companion  $\alpha \cdot \beta$  for all  $\alpha, \beta$  in  $\mathfrak{B}$

Denote by  $pder\mathfrak{B}$  the algebra of all pseudo-differentiations of  $\mathfrak{B}$  and  $ipder\mathfrak{B}$  the algebra of inner pseudo-differentiations of  $\mathfrak{B}$ .

The enlarged algebra of pseudo-differentiations of  $\mathfrak{B}$  is defined as  $Pder\mathfrak{B} = \{(D, z), D \in pder\mathfrak{B}, z \in Com(D)\}$  and the enlarged algebra of inner pseudo-differentiation is defined as  $Ipder\mathfrak{B} = \{(D, z), D \in ipder\mathfrak{B}, z \in Com(D)\}$ . Jos M. Prez-Isiquiro shows in [6] that, those algebras defined below are the Lie algebras with appropriate brackets.

**Definition 2.0.4.** Let  $\mathfrak{B}$  be a Bol algebra, and  $\mathcal{I}$  a subspace of  $\mathfrak{B}$ ,  $\mathfrak{B}$  is an ideal of  $\mathfrak{B}$  if  $\mathcal{I} \cdot \mathfrak{B} \subset \mathcal{I}$  and  $(\mathcal{I}; \mathfrak{B}, \mathfrak{B}) \subset \mathcal{I}$

**Proposition 2.0.1.** Let  $(\mathfrak{B}, (-; -, -), \cdot)$  be a Bol algebra,  $\mathcal{I}$  an ideal of  $\mathfrak{B}$ , then  $\mathfrak{B}/\mathcal{I}$  is a Bol algebra.

*Proof.* We know that,  $\mathfrak{B}/\mathcal{I}$  is an algebra. Define on  $\mathfrak{B}/\mathcal{I}$  the two operations:  $[x + \mathcal{I}, y + \mathcal{I}] = x \cdot y + \mathcal{I}$  and  $[x + \mathcal{I}; y + \mathcal{I}, z + \mathcal{I}] = (x; y, z) + \mathcal{I}$ . It is clear that,  $(\mathfrak{B}/\mathcal{I}, [-; -, -], [-, -])$  is a Bol algebra.  $\square$

**Proposition 2.0.2.** Let  $\mathfrak{B}_1$  and  $\mathfrak{B}_2$  be Bol Algebras,  $\varphi : \mathfrak{B}_1 \rightarrow \mathfrak{B}_2$  a morphism of Bol algebras. The kernel of  $\varphi$  is an ideal of  $\mathfrak{B}_1$  and the image of  $\varphi$  is a sub-Bol algebra of  $\mathfrak{B}_2$ .

*Proof.*  $\varphi : \mathfrak{B}_1 \rightarrow \mathfrak{B}_2$  is a morphism of Bol algebras.

We show that,  $[Ker\varphi, \mathfrak{B}] \subset Ker\varphi$  and  $[Ker\varphi; \mathfrak{B}, \mathfrak{B}] \subset Ker\varphi$ . Let  $x \in Ker\varphi$ ,  $y \in \mathfrak{B}_1$  and  $z \in \mathfrak{B}_1$ ; we have

$$\begin{aligned} \varphi([x, y]) &= [\varphi(x), \varphi(y)] \\ &= [0, \varphi(y)] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}
\varphi([x; y, z]) &= [\varphi(x); \varphi(y), \varphi(z)] \\
&= [0; \varphi(y), \varphi(z)] \\
&= 0
\end{aligned}$$

Next, we show that,  $[Im\varphi, Im\varphi] \subset Im\varphi$  and  $[Im\varphi; Im\varphi, Im\varphi] \subset Im\varphi$ .

Let  $a, b, c$  in  $Im\varphi$ , there exists  $x, y, z$  in  $\mathfrak{B}_1$  such that,  $a = \varphi(x)$ ,  $b = \varphi(y)$  and  $c = \varphi(z)$ . We have

$$\begin{aligned}
\varphi([a, b]) &= [\varphi(x), \varphi(y)] \\
&= [\varphi(x), \varphi(y)] \\
&= \varphi([x, y])
\end{aligned}$$

and

$$\begin{aligned}
\varphi([a; b, c]) &= [\varphi(x); \varphi(y), \varphi(z)] \\
&= [\varphi(x); \varphi(y), \varphi(z)] \\
&= \varphi([x; y, z])
\end{aligned}$$

Hence  $ker\varphi$  is an ideal of  $\mathfrak{B}_1$  and  $Im\varphi$  is a sub-Bol algebra of  $\mathfrak{B}_2$ .  $\square$

Let's remark here that the first isomorphism theorem remains unchanged for Bol algebras, i.e.,  $\mathfrak{B}_1/ker\varphi \cong Im\varphi$ .

**Proposition 2.0.3.** *Let  $(\mathfrak{B}, [-; -, -], [-, -])$  be a Bol algebra and  $X$  a subset of  $\mathfrak{B}$ . the intersection of all ideals containing  $X$  is the smallest ideal of  $\mathfrak{B}$  containing  $X$ , called the ideal generate by  $X$  and denoted  $\langle X \rangle$ .*

*Proof.*  $(\mathfrak{B}, [-; -, -], [-, -])$  is Bol algebra, and  $X$  a subset of  $\mathfrak{B}$ . Set  $\mathcal{I} = \bigcap_{i \in I} \mathcal{I}_i$  the intersection of all ideals of  $\mathfrak{B}$  containing  $X$ . Show that,  $\mathcal{I}$  is an ideal of  $\mathfrak{B}$ .

We have  $[\mathcal{I}, \mathfrak{B}, \mathfrak{B}] \subset \mathcal{I}_i$  for all  $i \in I$ , because  $\mathcal{I} \subset \mathcal{I}_i$ . Then, We have  $[\mathcal{I}, \mathfrak{B}, \mathfrak{B}] \subset \bigcap_{i \in I} \mathcal{I}_i$ ; and  $[\mathcal{I}, \mathfrak{B}, \mathfrak{B}] \subset \mathcal{I}$ . Likewise, we have  $[\mathcal{I}, \mathfrak{B}] \subset \mathcal{I}$ . In addition, we know that for all  $i \in I$ ,  $X \subset \mathcal{I}_i$ ; therefore,  $X \subset \mathcal{I}$ . Let,  $\mathcal{L}$  be another ideal containing  $X$ , it is clear that  $\mathcal{I} \subset \mathcal{L}$ .  $\square$

Bol algebras over a field  $K$  and their morphisms form a category denoted by  $Bal(K)$

**Proposition 2.0.4.** *The category  $Bal(K)$  is complete and cocomplete.*

*Proof.* We are going to show first that  $Bal(K)$  has products and equalizers of parallel pair of morphisms.

Let  $(\mathfrak{B}_i, [-; -, -]_i, [-, -]_i)_{i \in I}$  be a family of Bol algebras,  $I$  is a set.

We define on  $\prod_{i \in I} \mathfrak{B}_i$  the ternary operation  $[-; -, -]$  and the binary one  $[-, -]$  by:  
 $[(x_i); (y_i), (z_i)] = ([x_i; y_i, z_i])_{i \in I}$  and  $[(x_i), (y_i)] = ([x_i, y_i])_{i \in I}$ , for all  $(x_i)$ ,  $(y_i)$  and  $(z_i)$  in  $\prod_{i \in I} \mathfrak{B}_i$ . We recall that, the product of algebras is an algebra; it remains to prove the properties of binary and ternary operations. We have:

$$\begin{aligned} [(x_i), (x_i)] &= ([x_i, x_i]) \\ &= 0 \end{aligned}$$

The binary operation is linear and skew-symmetric. In other words,

$$\begin{aligned} [(x_i); (x_i), (y_i)] &= ([x_i; x_i, z_i]) \\ &= 0 \end{aligned}$$

It is easy to see that,  $[\cdot, \cdot]$  verifies the identity of Jaccobi and  $\mathfrak{D}_{(\alpha_i), (\beta_i)} : (x_i) \longrightarrow [(x_i); (\alpha_i), (\beta_i)]$  is a pseudo-differentiation with components  $(\alpha_i)$  and  $(\beta_i)$  on a binary operation  $[\cdot, \cdot]$ . The projections  $p_j : \prod_{i \in I} \mathfrak{B}_i \longrightarrow \mathfrak{B}_j$  for all  $i \in I$  are morphisms of Bol algebras.

Now, we construct the equalizer of parallel pair of  $Bol(K)$ . Let us consider  $f$  and  $g$  two morphisms of Bol algebra  $\mathfrak{B}_1$  in the Bol algebra  $\mathfrak{B}_2$ . Set

$E = \{x \in \mathfrak{B}_1 \mid f(x) = g(x)\}$ , it is clear that  $E$  is closed by the two operations  $[-; -, -]$  and  $[-, -]$ ; then  $(E, [-; -, -], [-, -])$  is a sub-Bol algebra. We define the inclusion map  $e : E \longrightarrow \mathfrak{B}_1$ .  $e$  is a morphism of Bol algebras which satisfies  $f \circ e = g \circ e$ . Let  $E'$  be another Bol algebra and  $e' : E' \longrightarrow \mathfrak{B}_1$  a morphism of Bol algebra which satisfies the equality  $f \circ e' = g \circ e'$ , show that there exists a unique  $\varphi : E' \longrightarrow E$  morphism of Bol algebras such that  $e \circ \varphi = e'$ ; see the diagram:

$$\begin{array}{ccccc} & & E & \xrightarrow{e} & \mathfrak{B}_1 & \xrightleftharpoons[f]{g} & \mathfrak{B}_2 \\ & \nearrow \psi & & & & & \\ E' & & & \searrow e' & & & \end{array}$$

We set for all  $x \in E'$ ,  $\varphi(x) = e'(x)$ .  $\varphi$  is a morphism of Bol algebra; and

$$\begin{aligned} e \circ \varphi(x) &= e(e'(x)) \\ &= e'(x). \end{aligned}$$

Let  $\psi : E' \longrightarrow E$  be another morphism of Bol algebra which has the same properties as  $\varphi$ . we have,

$$\begin{aligned}\psi(x) &= e(\varphi(x)) \\ &= \varphi(x).\end{aligned}$$

Then  $\psi = \varphi$ .

We set now  $\coprod_{i \in I} \mathfrak{B}_i = \{(x_i), i \in I_0\}$  where  $I_0$  is a finite subset of  $I$ . We

define same as for products, the ternary operation  $[-; -, -]$  and the binary operation  $[-, -]$ . It is easy to see that,  $(\coprod_{i \in I} \mathfrak{B}_i, [-; -, -], [-, -])$  is a

Bol algebra, named the coproduct of the family  $(\mathfrak{B}_i, [-; -, -]_i, [-, -]_i)_{i \in I}$ .

Now construct the coequalizer of morphisms  $f$  and  $g$  defined above. We know that,  $Imf + Img$  is sub-Bol space of Bol algebra  $\mathfrak{B}_1$ ; But not necessarily an ideal of Bol algebra. According to proposition (2.0.3), let us consider  $\mathcal{I} = \langle Imf + Img \rangle$  to be a smallest ideal generated by  $Imf + Img$ .  $\mathfrak{B}/\mathcal{I}$  is a Bol algebra by proposition(2.0.1). Let us show that,  $(\mathfrak{B}/\mathcal{I}, p)$  is a coequalizer of  $f$  and  $g$ , where  $p$  is the canonical projection.

Let  $x \in \mathfrak{B}_1$ ,

$$\begin{aligned}p \circ f(x) &= f(x) + \mathcal{I} \\ &= \mathcal{I} \\ &= p \circ g(x)\end{aligned}$$

We have  $p \circ f = p \circ g$ . Let  $(Q, h)$  be another pair, with  $Q$  a Bol algebra and  $h : \mathfrak{B}_2 \longrightarrow Q$  a morphism of Bol algebras such that  $h \circ f = h \circ g$ . We show that, there exists a unique morphism of Bol algebras  $\varphi : \mathfrak{B}/\mathcal{I} \longrightarrow \mathfrak{B}_2$  such that  $\varphi \circ p = h$ .

we define  $\varphi : \mathfrak{B}/\mathcal{I} \longrightarrow Q$  such that,  $\varphi(x + \mathcal{I}) = h(x)$ ;  $\varphi$  is a morphism of Bol algebras.

Let  $\psi : \mathfrak{B}/\mathcal{I} \longrightarrow Q$  be another morphism of Bol algebras such that  $\psi \circ p = h$ . We have for all  $x \in \mathfrak{B}_2$ ,

$$\begin{aligned}\psi(x + \mathcal{I}) &= \psi \circ p(x) \\ &= \varphi \circ p(x) \\ &= \varphi(x + \mathcal{I})\end{aligned}$$

Then  $\varphi$  is unique.  $(\mathfrak{B}/\mathcal{I}, p)$  is a coequalizer of the pair  $f, g$ . Hence  $Bal(K)$  is complete and cocomplete see [13]

□

### 3. REPRESENTATION OF BOL ALGEBRAS

Before constructing the representation of Bol algebra, we must first remark that it is not done in an analogous manner to that of the

Lie triple systems. In [11] W. Bertrand and M. Didry defined a representation of Lie triple system  $T$ , in  $\text{End}(V)$  by giving two bilinear maps:

$$\begin{aligned} \mathfrak{D} : T \times T &\longrightarrow \text{End}(V) \\ (\alpha, \beta) &\longmapsto \mathfrak{D}_{\alpha, \beta} \end{aligned}$$

$$\begin{aligned} \Delta : T \times T &\longrightarrow \text{End}(V) \\ (\alpha, \beta) &\longmapsto \Delta_{\alpha, \beta} \end{aligned}$$

Which satisfy certain properties; where  $V$  is a vector space called  $T$ -module. Talking of Bol algebras, one must remark that the presence of the second binary operation brings about supplementary conditions which may not be fulfilled by the above maps.

Let  $(\mathfrak{B}, (;, ), (;))$  be a Bol algebra and  $V$  be a vector space over a field  $K$ , we consider the following bilinear maps as:

$$\begin{aligned} \cdot : V \times \mathfrak{B} &\longrightarrow V \\ (v, \alpha) &\longmapsto v \cdot \alpha \end{aligned}$$

$$\begin{aligned} \cdot : \mathfrak{B} \times V &\longrightarrow V \\ (\alpha, v) &\longmapsto \alpha \cdot v \end{aligned}$$

Likewise, we define the trilinear maps :

$$\begin{aligned} [-; -, -] : V \times \mathfrak{B} \times \mathfrak{B} &\longrightarrow V \\ (v, \alpha, \beta) &\longmapsto [v; \alpha, \beta] \end{aligned}$$

$$\begin{aligned} [-; -, -] : \mathfrak{B} \times V \times \mathfrak{B} &\longrightarrow V \\ (\alpha, v, \beta) &\longmapsto [\alpha; v, \beta] \end{aligned}$$

$$\begin{aligned} [-; -, -] : \mathfrak{B} \times \mathfrak{B} \times V &\longrightarrow V \\ (\alpha, \beta, v) &\longmapsto [\alpha; \beta, v] \end{aligned}$$

**Definition 3.0.5.** *A vector space  $V$  is a Bol module if for all  $\alpha, \beta, \gamma, \tau$  in  $\mathfrak{B}$  and  $v$  in  $V$ , the following properties are true:*

- (1)  $\alpha \cdot v = -v \cdot \alpha$
- (2)  $[v; \alpha, \beta] + [v; \beta, \alpha] = 0$
- (3)  $[v; \alpha, \beta] + [\alpha; v, \beta] + [\alpha; \beta, v] = 0$
- (4)  $[(\alpha; \beta, \gamma); v, \tau] = [\alpha; \beta, [\gamma; v, \tau]] + [\alpha; [\beta, v, \tau], \gamma] + [[\alpha; v, \tau]; \beta, \gamma]$
- (5)  $[(\alpha; \beta, \gamma), v \cdot \tau] = v \cdot (\alpha; \beta, \tau) + [v; \alpha, \beta] \cdot \tau + [(\beta; \alpha); \tau, v] + (v \cdot \tau) \cdot (\beta; \alpha)$

The operations defined above are called the  $\mathfrak{B}$ -actions or actions of  $\mathfrak{B}$  on  $V$ . Actually, the Bol module defined above is called left Bol module.

**Example 3.0.2.** *Let  $(\mathfrak{B}, (-; -, -), (-; -))$  be a Bol algebra,  $\mathfrak{B}$  seen as a vector space is a Bol module.*

We remark here that, the notion of Bol module is not the analogous notion defined by Terrel L.Hodge and Brian J.Parshall for Lie triple systems [10]. According to them, a vector space  $V$  is a  $\mathfrak{B}$ -module provided that,  $E_V = \mathfrak{B} \oplus V$  possesses a structure of Bol algebra such that: (a)  $\mathfrak{B}$  is a sub-Bol algebra of  $E_V$ , (b)  $V$  is an abelian ideal of  $E_V$ , (c)  $[V; V, \mathfrak{B}] = [V; \mathfrak{B}, V] = [\mathfrak{B}; V, V] = 0$

**Proposition 3.0.5.** *Every  $\mathfrak{B}$ -module  $V$  is a Bol module.*

*Proof.* Let  $V$  be a  $\mathfrak{B}$ -module,  $E_V = \mathfrak{B} \oplus V$  has a structure of Bol algebra with brackets  $[-; -, -]$  and  $[-, -]$ .

We define the bilinear maps:

$$\begin{aligned} [-, -]_1 : V \times \mathfrak{B} &\longrightarrow V \\ (v, \alpha) &\longmapsto [v, \alpha] \end{aligned}$$

$$\begin{aligned} [-, -]_1 : \mathfrak{B} \times V &\longrightarrow V \\ (\alpha, v) &\longmapsto [\alpha, v] \end{aligned}$$

Likewise, we define the trilinear maps :

$$\begin{aligned} [-; -, -]_1 : V \times \mathfrak{B} \times \mathfrak{B} &\longrightarrow V \\ (v, \alpha, \beta) &\longmapsto [v; \alpha, \beta] \end{aligned}$$

$$\begin{aligned} [-; -, -]_1 : \mathfrak{B} \times V \times \mathfrak{B} &\longrightarrow V \\ (\alpha, v, \beta) &\longmapsto [\alpha; v, \beta] \end{aligned}$$

$$\begin{aligned} [-; -, -]_1 : \mathfrak{B} \times \mathfrak{B} \times V &\longrightarrow V \\ (\alpha, \beta, v) &\longmapsto [\alpha; \beta, v] \end{aligned}$$

The properties (1) – (5) of definition of a Bol module hold. □

**Definition 3.0.6.** *Every linear map between two Bol modules  $V$  and  $W$  which preserve the  $\mathfrak{B}$ -actions is call morphism of Bol modules.*

**Example 3.0.3.**

*Let  $V$  be a Bol module, the identity map of  $V$  is a morphism of Bol module.*

*$V$  is a Bol module,  $l_\tau : V \longrightarrow V$  defined by  $L_\tau(v) = v \cdot \tau$ , is a morphism of Bol module.*



Before studying the general representation of Bol algebras, we are going to investigate a particular case of such representation that N.Jacobson called regular representation in the case of Jordan Algebras [17].

$V$  is a Bol module, Let's consider a linear map  $l_\tau : V \longrightarrow V$  defined by  $L_\tau(v) = v \cdot \tau$ .  $L_\tau$  is an endomorphism of Bol module  $V$  into itself, which we call left transformation. we can not define a right transformation here because , every right transformation  $R_\tau(v) = \tau \cdot v$  is also left transformation as:

$$\begin{aligned} R_\tau(v) &= \tau \cdot v \\ &= -v \cdot \tau \\ &= L_{-\tau}(v) \end{aligned}$$

We define two operations in left transformations by:

for all  $\alpha, \beta, \gamma \in \mathfrak{B}$  and  $v \in V$ ,  $[L_\alpha, L_\beta](v) = L_{(\alpha; \beta)}(v)$  and  $[L_\alpha; L_\beta, L_\gamma](v) = L_{(\alpha; \beta, \gamma)}(v)$ .

Now, we consider the map

$$\begin{aligned} \rho : \mathfrak{B} &\longrightarrow \text{End}(V) \\ \tau &\longmapsto L_\tau \end{aligned}$$

$\rho$  is a morphism of Bol algebras.  $\rho$  is a representation of Bol algebras, called regular representation.

Let us construct the general representation of Bol algebras.

Let  $V$  be a Bol module, we define the following maps:

$$\begin{aligned} m(\alpha, \beta) : V &\longrightarrow V \\ v &\longmapsto [\alpha; \beta, v] \end{aligned}$$

$$\begin{aligned} c(\alpha, \beta) : V &\longrightarrow V \\ v &\longmapsto [\alpha; v, \beta] \end{aligned}$$

$$\begin{aligned} r(\alpha, \beta) : V &\longrightarrow V \\ v &\longmapsto [v; \alpha, \beta] \end{aligned}$$

$$\begin{aligned} L_\tau : V &\longrightarrow V \\ v &\longmapsto \tau \cdot v \end{aligned}$$

$$\begin{aligned} R_\tau : V &\longrightarrow V \\ v &\longmapsto v \cdot \tau \end{aligned}$$

We have the following properties:

**lemma 3.0.1.** *The following properties are satisfied:*

- (p<sub>1</sub>)  $R_\tau = L_{-\tau}$
- (p<sub>2</sub>)  $m(\alpha, \beta) + r(\alpha, \beta) = 0$
- (p<sub>3</sub>)  $m(\alpha, \beta) + c(\alpha, \beta) + r(\alpha, \beta) = 0$
- (P<sub>4</sub>)  $c(\mathfrak{D}_{\alpha, \beta}(\gamma), \tau) = m(\alpha, \beta) \circ c(\gamma, \tau) + c(\alpha, \gamma) \circ c(\beta, \tau) + r(\beta, \gamma) \circ c(\alpha, \tau)$
- (p<sub>5</sub>)  $m(\alpha, \beta) \circ L_\tau = L_{(\mathfrak{D}_{\alpha, \beta})(\tau)} + L_\tau \circ r(\alpha, \beta) + m((\beta; \alpha), \tau) + L_{(\beta; \alpha)} \circ L_\tau$

*Proof.* (p<sub>1</sub>) was shown in regular representation.

For (p<sub>2</sub>), (p<sub>3</sub>), (p<sub>4</sub>), we write the relations (1) and (2) in the definition of Bol module in the sense of  $m(\alpha, \beta)$ ,  $c(\alpha, \beta)$  and  $r(\alpha, \beta)$ .

For (p<sub>5</sub>), we translate the relation (5) in the definition by the morphism  $m(\alpha, \beta), r(\alpha, \beta), L_\tau$  and  $\mathfrak{D}_{\alpha, \beta}(\tau)$ .  $\square$

**Definition 3.0.7.** *A general representation of a Bol algebra  $\mathfrak{B}$  in  $End(V)$ , the space of endomorphisms of a Bol module  $V$ , is given by two linear maps:*

$$\begin{array}{ccc} L_\tau : \mathfrak{B} & \longrightarrow & End(V) \\ \tau & \longmapsto & L_\tau \end{array}$$

$$\begin{array}{ccc} R : \mathfrak{B} & \longrightarrow & End(V) \\ \tau & \longmapsto & R_\tau \end{array}$$

and tree bilinear maps:

$$\begin{array}{ccc} m : \mathfrak{B} \times \mathfrak{B} & \longrightarrow & End(V) \\ (\alpha, \beta) & \longmapsto & m(\alpha, \beta) \end{array}$$

$$\begin{array}{ccc} c : \mathfrak{B} \times \mathfrak{B} & \longrightarrow & End(V) \\ (\alpha, \beta) & \longmapsto & c(\alpha, \beta) \end{array}$$

$$\begin{array}{ccc} r : \mathfrak{B} \times \mathfrak{B} & \longrightarrow & End(V) \\ (\alpha, \beta) & \longmapsto & r(\alpha, \beta) \end{array}$$

such that (p<sub>1</sub>)-(p<sub>5</sub>) hold.  $V$  is called module of representation.

We can replace  $End(V)$  in the definition above by a unitary associative algebra  $A$ .

**Proposition 3.0.6.** *Let  $V$  be a Bol module, for all  $\alpha, \beta, \gamma$  in  $\mathfrak{B}$ , the following equality holds:*

$$\begin{aligned} & m(\alpha, \beta) \circ m(\gamma, \tau) + m(\alpha, \beta) \circ r(\gamma, \tau) + r(\beta, \alpha) \circ m(\alpha, \tau) + r(\beta, \gamma) \circ r(\alpha, \tau) = \\ & m(\mathfrak{D}_{\alpha, \beta}(\gamma), \tau) + r(\mathfrak{D}_{\alpha, \beta}(\gamma), \tau) + m(\alpha, \gamma) \circ m(\beta, \tau) + m(\alpha, \gamma) \circ r(\beta, \tau) + \\ & r(\alpha, \gamma) \circ m(\beta, \tau) + r(\alpha, \gamma) \circ r(\beta, \tau). \end{aligned}$$

*Proof.* We use the property  $(p_4)$  of lemma 3.0.1 and we replace  $c(\alpha, \beta)$  by its value  $c(\alpha, \beta) = -m(\alpha, \beta) - r(\alpha, \beta)$  in the identity  $(p_2)$  of lemma 3.0.1  $\square$

**Proposition 3.0.7.** *Let  $\mathfrak{B}$  be a Bol algebra over a field  $k$ , and  $V$  a Bol module. If the extension  $E_V = \mathfrak{B} \oplus V$  has a structure of Bol algebra, then  $\mathfrak{B}$  is a sub-Bol algebra of  $E_V$  and  $V$  is an ideal of  $E_V$ .*

*Proof.* Let  $P : E_V \longrightarrow \mathfrak{B}$  be the projection on  $\mathfrak{B}$  with direction  $V$ .  $P$  is a morphism of Bol algebras;  $V = \ker P$  and  $\mathfrak{B} = \text{im} P$ , according to proposition 2.02,  $V$  is an ideal of  $E_V$  and  $\mathfrak{B}$  is a sub-Bol algebra of  $E_V$ .  $\square$

**Proposition 3.0.8.** *If  $(V, L, R, m, c, r)$  and  $(W, L', R', m', c', r')$  are two representations of Bol algebras  $\mathfrak{B}$ , then  $(V \oplus W, L \oplus L', R \oplus R', m \oplus m', c \oplus c', r \oplus r')$  is again a representation of  $\mathfrak{B}$ , called a direct sum.*

*Proof.* we define the following linear maps:

$$(L \oplus L')_\tau : \begin{array}{ccc} V \oplus W & \longrightarrow & V \oplus W \\ v \oplus w & \longmapsto & L_\tau(v) \oplus L'_\tau(w) \end{array}$$

$$(R \oplus R')_\tau : \begin{array}{ccc} V \oplus W & \longrightarrow & V \oplus W \\ v \oplus w & \longmapsto & R_\tau(v) \oplus R'_\tau(w) \end{array}$$

$$(m \oplus m')(\alpha, \beta) : \begin{array}{ccc} V \oplus W & \longrightarrow & V \oplus W \\ v \oplus w & \longmapsto & m_\tau(v) \oplus m'_\tau(w) \end{array}$$

$$(c \oplus c')(\alpha, \beta) : \begin{array}{ccc} V \oplus W & \longrightarrow & V \oplus W \\ v \oplus w & \longmapsto & c(\alpha, \beta)(v) \oplus c'(\alpha, \beta)(w) \end{array}$$

$$(r \oplus r')(\alpha, \beta) : \begin{array}{ccc} V \oplus W & \longrightarrow & V \oplus W \\ v \oplus w & \longmapsto & r(\alpha, \beta)(v) \oplus r'(\alpha, \beta)(w) \end{array}$$

It is clear that properties  $(p_1) - (p_5)$  of lemma 3.0.1 hold.  $\square$

#### 4. DUAL REPRESENTATION AND DUALITY PRINCIPLE

**4.1. The dual representation.** Let  $(V, L, R, m, c, r)$  be a representation of Bol algebra, we wish to define a structure of Bol module in the dual space  $V^*$ . We set:

$$m^*(\alpha, \beta) : \begin{array}{ccc} V^* & \longrightarrow & V^* \\ f & \longmapsto & m^*(\alpha, \beta)(f) \end{array}$$

with

$$\begin{array}{ccc} m^*(\alpha, \beta)(f) : & V & \longrightarrow K \\ & v & \longmapsto f([\beta; \alpha, v]) \end{array}$$

$$\begin{array}{ccc} c^*(\alpha, \beta) : & V^* & \longrightarrow V^* \\ & f & \longmapsto c^*(\alpha, \beta)(f) \end{array}$$

with

$$\begin{array}{ccc} c^*(\alpha, \beta)(f) : & V & \longrightarrow K \\ & v & \longmapsto f([\beta; v, \alpha]) \end{array}$$

$$\begin{array}{ccc} r^*(\alpha, \beta) : & V^* & \longrightarrow V^* \\ & f & \longmapsto r^*(\alpha, \beta)(f) \end{array}$$

with

$$\begin{array}{ccc} r^*(\alpha, \beta)(f) : & V & \longrightarrow K \\ & v & \longmapsto f([v; \beta, \alpha]) \end{array}$$

$$\begin{array}{ccc} L_\tau^* : & V^* & \longrightarrow V^* \\ & f & \longmapsto L^*(\tau)(f) \end{array}$$

with

$$\begin{array}{ccc} L^*(\tau)(f) : & V & \longrightarrow K \\ & v & \longmapsto f(L_\tau(v)) \end{array}$$

$$\begin{array}{ccc} R_\tau^* : & V^* & \longrightarrow V^* \\ & f & \longmapsto R^*(\tau)(f) \end{array}$$

with

$$\begin{array}{ccc} R^*(\tau)(f) : & V & \longrightarrow K \\ & v & \longmapsto f(R_\tau(v)) \end{array}$$

The properties (1) – (5) of the definition of Bol module are satisfied. Hence,  $V^*$  is a Bol module called the dual module of  $V$ .  $(V^*, L^*, R^*, m^*, c^*, r^*)$  is also a representation of Bol module  $\mathfrak{B}$  that we call dual representation. Note that, if  $V$  is a right Bol module,  $V^*$  is a left Bol Module. If  $A : V \longrightarrow V$  is an operator in  $V$ , its dual operator  $A^*$  is defined as follows:

$$\begin{array}{ccc} A^* : & V^* & \longrightarrow V^* \\ & f & \longmapsto f \circ A \end{array}$$

**4.2. The duality principle.** We know that, for finite dimensional module  $V$  over a field,  $V$  is the dual of its dual module  $V^*$ .

More generally, we can define for any general representation of Bol algebra  $\mathfrak{B}$  in an algebra  $A$  its opposite representation in the algebra  $A^{op}$  by setting:

$$r^{op}(\alpha, \beta) = r(\beta, \alpha)$$

$$c^{op}(\alpha, \beta) = c(\beta, \alpha)$$

$$m^{op}(\alpha, \beta) = m(\beta, \alpha); L_\tau^{op} = R_\tau \text{ and } R_\tau^{op} = L_\tau.$$

As above it is seen that this is again a representation. As an application of this, we get a duality principle similar to the one of Jordan pairs formulated by O.Loos[18] and another for Lie triple systems formulate by W.Bertrand and M.Didry [11].

**Theoreme 4.2.1.** *If  $I$  is an identity in  $L_\tau, R_\varsigma, r(\alpha, \beta), c(X, Y)$  and  $m(S, T)$  valid for all left Bol algebras over  $\mathbb{R}$ , Then its dual identity  $I^*$ , obtained by replacing  $L_\tau$  by  $R_\tau$ ;  $R_\varsigma$  by  $L_\varsigma$ ;  $r(\alpha, \beta)$  by  $r(\beta, \alpha)$ ;  $c(X, Y)$  by  $c(Y, X)$  and  $m(S, T)$  by  $m(T, S)$  and reversing the order of all factors, is also valid for all right Bol algebras over  $\mathbb{R}$ .*

*Proof.* We remark that, if  $A$  is the representation algebra of Bol algebra  $\mathfrak{B}$ ,  $A^{op}$  is the representation algebra of  $\mathfrak{B}^{op}$ . We denote by  $Bal_g$  the category of left Bol algebras over  $\mathbb{R}$  and by  $Bal_d$  the category of right Bol algebras over  $\mathbb{R}$ . We define the functor:

$$\begin{array}{ccc} \mathcal{F} : & Bal_d & \longrightarrow Bal_g \\ & \mathfrak{B} & \longmapsto \mathfrak{B}^{op} \end{array}$$

Let  $(\mathfrak{B}, (-; -, -), (-; -))$  be a right Bol algebra; from right Bol algebra  $\mathfrak{B}$  we can obtain left Bol algebra  $\mathfrak{B}^{op}$  by considering  $\mathfrak{B}$  with the operations

$$[x; y, z] = -(x; y, z) \text{ and } [x, y] = -x \cdot y.$$

The functor  $\mathcal{F}$  realizes an isomorphism of category; and the category of right Bol modules is equivalent to a category of left Bol modules.  $\square$

One can embed,  $\mathfrak{B}$  into a Lie algebra  $\mathfrak{G}$  as;  $\mathfrak{G} = \mathfrak{B} \oplus \mathfrak{h}$  where,  $\mathfrak{h}$  is a Lie subalgebra of  $\mathfrak{G}$ . and we have the short sequence  $\mathfrak{h} \rightarrow \mathfrak{G} \rightarrow \mathfrak{G}/\mathfrak{h} \simeq \mathfrak{B}$ . The operation in  $\mathfrak{B}$  from the Lie algebra  $\mathfrak{G}$  is obtained by means of projection on  $\mathfrak{B}$  parallel to  $\mathfrak{h}$  that is

$$\begin{array}{ccc} \mathfrak{G} \times \mathfrak{G} & \longrightarrow & \mathfrak{G} \\ (\xi, \eta) & \longrightarrow & [\xi, \eta] \\ \downarrow & & \\ \Pi_{[\xi, \eta]} & = & \xi \cdot \eta \end{array}$$

We have  $0 \rightarrow \mathfrak{h} \leftrightarrow \mathfrak{G} \rightarrow \mathfrak{G}/\mathfrak{h} \simeq \mathfrak{B} \rightarrow 0$  which is the short exact sequence of  $\mathfrak{h}$ -module. Therefor we have the following proposition

**Proposition 4.2.1.** *If  $\mathfrak{B}$  is a Bol algebra with the operation  $(-\cdot-)$ ,  $(-;-,-)$ . Then the set  $\mathfrak{G} = \mathfrak{B} \oplus \wedge^2 \mathfrak{B}$  is the enveloping Lie algebra for the Bol algebra  $\mathfrak{B}$ .*

**Proof** Set  $[x, y]_{\mathfrak{G}} = x \cdot y + x \wedge y$  and from this, the Jacobi identity follows.

Another way of proving this theorem is to consider the set  $Pder\mathfrak{B}$ , of the pseudo derivation of  $\mathfrak{B}$  and  $IPder\mathfrak{B}$ , the set of inner pseudo-differentiation of  $\mathfrak{B}$ . As it is shown in [12], the first is the maximal standard enveloping Lie algebra while second is the standard minimal enveloping Lie algebra. Hence the proof.

**Proposition 4.2.2.** [7] *Any pair of Lie algebras  $(\mathfrak{G}, \mathfrak{h})$  such that  $\mathfrak{G} = \mathfrak{B} \oplus \mathfrak{h}$ ,  $[\mathfrak{B}, [\mathfrak{B}, \mathfrak{B}]] \subset \mathfrak{B}$ ,  $[\mathfrak{B}, \mathfrak{B}] \cap \mathfrak{B} = \{0\}$  defines on  $\mathfrak{B}$  a Bol algebra with the operations  $\xi \cdot \eta = \square[\xi, \eta]$ , the projection on  $\mathfrak{B}$  parallel to  $\mathfrak{h}$ , and  $(\zeta; \xi, \eta) = [\zeta, [\xi, \eta]]$ ,  $\forall \zeta, \xi, \eta \in \mathfrak{B}$ .*

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DEPARTMENT OF MATHEMATICS, University of Yaounde I, P.O.Box:  
812 Yaounde-Cameroon.

E-mail: ndouneko@yahoo.fr

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES,  
POLYTECHNIC, Yaounde, P.O.Box: 8390 Yaounde-Cameroon.

E-mail: tbouetou@yahoo.fr